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On the influence of a bimaterial interface on dynamic stress intensity factors

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SUMMARY
The method of dynamic Green's function and the integral transforms are applied to investigate the elastodynamic stress intensity factor of a crack straddling an interface of a bimaterial composite. The crack which extends to infinity on one side is assumed to extend an arbitrary distance a on the other side of the interface. Anti-plane line loads are suddenly applied at time $t = 0$ on either side of the crack surface at arbitrary distances l_1 and l_2 from the interface. The effect of the interface on the dynamic stress field near the crack tip is studied. It is found that the transmitted wave through the interface and reflected wave from the interface serve to increase or decrease the stress field in the vicinity of the crack tip depending on the elastic properties of the two materials.

1. Introduction

Composites are increasingly replacing conventional single materials in industrial use. Though they are made to tolerate a wide variety of applied loads, the behavior of composites containing flaws or cracks which extend from one medium to the other through the interface is not well known and the study of the stress response in this case is of great practical significance. The quantity of theoretical interest is the stress field in the vicinity of flaws or cracks. Appropriate quasistatic models of cracks in composites have already been proposed by some authors, see for example Refs. [1, 2]. However, the influence of the bimaterial interface for the corresponding dynamic problems has not been fully explored.

Recently, the elastodynamic problem of cracks extending from the interface into materials of a bimaterial composite is studied by Atkinson [3]. He assumed that the cracks propagate perpendicular to the interface on both sides with constant but different velocities. Since there is no characteristic length involved in the problem, the method of homogeneous solution using the self-similarity nature of certain field variables has been employed in Ref. [3]. However, if a crack already straddles an interface of a bimaterial composite and is subjected to dynamic loads, the method of homogeneous solution cannot be applied since the diffracted waves will have more than one center. Such problems which involve characteristic length can be solved by a method based on dynamic Green's function. The method was first used by Evvard [4] and has been referred to as the method of Evvard or the dynamic Green's function method, and has been subsequently employed by Kostrov [5] and Achenbach [6, 7] for problems of crack propagation in single material. The method is also briefly discussed in Ref. [8]. The applicability of this method for cracks in composite elastic medium has been explored recently in Ref. [9]. Some elastodynamic problems of forced transient motions and interface failure using this method have been studied by Chilton [10].

The method of dynamic Green's function is based on solving the wave equation where the displacement gradient is generated by an antiplane impulsive line load applied at some point in the medium. Then, according to the standard Green's function theory [11], the displacement field may be found by superposing this fundamental solution with the applied traction. The solution will be obtained in the form of an integral equation. In many cases these resulting integral equations are very difficult to solve while in some cases, they can be conveniently reduced to integral equations of Abel's type. The procedure for the latter case is shown in the appendix.

In this paper, we find the elastodynamic stress intensity factor for a crack existing in a bimaterial composite of two perfectly joined elastic half spaces of different elastic properties. The model chosen is shown in Fig. 1. The crack is of semi-infinite extent in one material and penetrates perpendicular to the interface into the second material to a distance a. The composite is subjected to suddenly applied loads, which generate wave motion. An interesting question to investigate is whether the interface causes an increase or decrease in the dynamic stress intensity factor at the crack tip at $x = a$. It may be anticipated that this will depend crucially on the ratios of the properties of the two materials.

Instead of analyzing the elastodynamic fields for loads applied in the medium, with a stress free crack, we analyze the inverse case where loads are applied on the faces of the crack. The two cases are related by superposition considerations. In particular, we will determine the elastodynamic stress-intensity factors at the crack tip for the case that anti-plane equal and opposite concentrated line loads are applied to the faces at $x = -l_1$ and at $x = l_2$, see Fig. 1. These anti-plane disturbances generate horizontally polarized wave

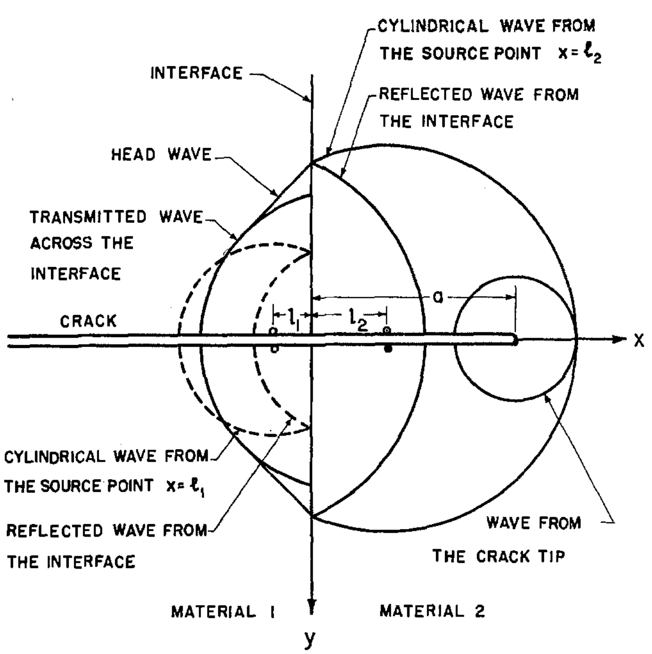

Figure 1. Pattern of wavefronts and the position of the crack tip.

motion in the medium. In Fig. 1, the wavefronts are drawn for the case $\mu_2 > \mu_1$, where μ_i ($i = 1, 2$) are the shear moduli of materials 1 and 2, respectively.

2. Formulation of the problem

The two equal and opposite loads applied at $x = -l_1$ and $x = l_2$ at time $t = 0$ to the upper and lower surfaces of the crack (see Fig. 1), generate an antiplane displacement field which will be antisymmetric relative to the plane $y = 0$. As a consequence the displacement w vanishes for $x \geqq a$, i.e.,

$$x \geqq a: \quad w(x, 0, t) = 0. \tag{1}$$

On the crack surfaces the concentrated loads give rise to the following boundary conditions for $t > 0$

$$\tau_{yz} = \begin{cases} \tau_0 \delta(x + l_1)t \\ \tau_0 \delta(x - l_2)t \end{cases} \tag{2}$$

where $\delta(\cdot)$ is the Dirac delta function. Equations (1) and (2) are boundary conditions for either the half-space $y \geqq 0$ or the half-space $y \leqq 0$.

Let us consider the half-space $y \geqq 0$. This half-space consists of two quarter spaces of different elastic solids, which are perfectly joined along the common boundary defined by $x = 0$, see Fig. 1. We use Cartesian coordinates x, y, z, where z is the out-of-plane coordinate. We denote the material properties in the regions $x \leqq 0$ and $x \geqq 0$ by subscripts 1 and 2 respectively.

The shear tractions (2) generate horizontally polarized shear motion in the z-direction only. The displacements w_1 and w_2 in the z-direction are governed by the wave equations

$$\nabla^2 w_1 = (1/c_1^2)\partial^2 w_1/\partial t^2, \qquad \nabla^2 w_2 = (1/c_2^2)\partial^2 w_2/\partial t^2 \tag{3a, b}$$

for $x \leqq 0$ and $x \geqq 0$ respectively. In equations (3a, b) ∇^2 denotes the Laplacian operator in the xy-plane and

$$c_1 = (\mu_1/\rho_1)^{\frac{1}{2}} \text{ and } c_2 = (\mu_2/\rho_2)^{\frac{1}{2}} \tag{4a, b}$$

are the shear wave speeds, where μ_1, μ_2 and ρ_1 and ρ_2 represent shear moduli and mass densities in the solids 1 and 2, respectively. The relevant shear stresses are

$$(\tau_{yz})_i = \mu_i \partial w_i/\partial y; \qquad (\tau_{xz})_i = \mu_i \partial w_i/\partial x; \qquad i = 1, 2. \tag{5a, b}$$

The problem at hand thus consists of finding solutions of equations (3a, b) satisfying the conditions (1) and (2), as well as the following conditions imposing continuity of the displacement and the stress at $x = 0$, $y \geqq 0$:

$$w_1 = w_2; \qquad \mu_1 \partial w_1/\partial x = \mu_2 \partial w_2/\partial x. \tag{6a, b}$$

In addition, the following initial conditions must be satisfied

$$t < 0: \quad w_i(x, y, t) = \partial w_i(x, y, t)/\partial t = 0. \tag{7}$$

3. Displacement fields in the composite half-space

Let us first investigate the nature of the displacement fields in the composite half-space $y \geqq 0$ for the case that the surface $y = 0$ is subjected solely to a concentrated anti-plane load (see Ref. [9]). No generality is lost if we assume that the load is applied at $x = l_2$, where $l_2 > 0$. The boundary condition on $y = 0$, $-\infty < x < \infty$ then is

$$(\tau_{yz})_2 = \mu_2 \partial w_2/\partial y = \tau_0 \delta(x - l_2) t H(t). \tag{8}$$

For the boundary conditions prescribed above, it is convenient to apply cosine transform over the variable y and Laplace transform over time t. These transforms are defined as

$$g^*(x, \gamma, t) = \int_0^\infty g(x, y, t) \cos \gamma y \, dy, \tag{9a}$$

$$\bar{g}(x, y, p) = \int_0^\infty g(x, y, t) \mathrm{e}^{-pt} dt. \tag{9b}$$

Applying first the cosine transform (9a) and then the Laplace transform (9b) and using the initial conditions (7) in Eq. (3a), we obtain for $x \leqq 0$

$$d^2 \bar{w}_1^*/dx^2 - (\gamma^2 + a_1^2 p^2) \bar{w}_1^* = 0. \tag{10}$$

Equation (10) is an ordinary differential equation, which can easily be solved to yield

$$\bar{w}_1^* = A_1(\gamma) \exp[(\gamma^2 + a_1^2 p^2)^{\frac{1}{2}} x]. \tag{11}$$

Similarly, for medium 2 the solution satisfying the boundary condition (8) is obtained as

$$\bar{w}_2^* = A_2(\gamma) \exp[-(\gamma^2 + a_2^2 p^2)^{\frac{1}{2}} x] - \frac{\tau_0}{2\mu_2} \frac{1}{p^2} \frac{\exp[-(\gamma^2 + a_2^2 p^2)^{\frac{1}{2}} |x - l_2|]}{(\gamma^2 + a_2^2 p^2)^{\frac{1}{2}}} \tag{12}$$

where A_1 and A_2 are functions of γ to be determined from the continuity conditions (6a) and (6b), and a_1 and a_2 are slownesses defined by

$$a_i = 1/c_i; \quad (i = 1, 2). \tag{13a, b}$$

Applying cosine and Laplace transforms to Eqs. (6a, b) and then employing Eqs. (11) and (12), continuity of displacement at $x = 0$ gives

$$A_1(\gamma) - A_2(\gamma) = -\frac{\tau_0}{2\mu_2} \frac{1}{p^2} \frac{\exp[-(\gamma^2 + a_2^2 p^2)^{\frac{1}{2}} l_2]}{(\gamma^2 + a_2^2 p^2)^{\frac{1}{2}}}, \tag{14}$$

while the continuity of shear stress yields

$$A_1(\gamma) = -\frac{\mu_2}{\mu_1} \frac{(\gamma^2 + a_2^2 p^2)^{\frac{1}{2}}}{(\gamma^2 + a_1^2 p^2)^{\frac{1}{2}}} A_2(\gamma) - \frac{\tau_0}{2\mu_1} \frac{1}{p^2} \frac{\exp[-(\gamma^2 + a_2^2 p^2)^{\frac{1}{2}} l_2]}{(\gamma^2 + a_1^2 p^2)^{\frac{1}{2}}}. \tag{15}$$

From (14) and (15) we obtain

$$A_1(\gamma) = \frac{\tau_0}{p^2} \frac{\exp[-(\gamma^2 + a_2^2 p^2)^{\frac{1}{2}} l_2]}{\mu_1 (\gamma^2 + a_1^2 p^2)^{\frac{1}{2}} + \mu_2 (\gamma^2 + a_2^2 p^2)^{\frac{1}{2}}} \tag{16}$$

and

$$A_2(\gamma) = \frac{\tau_0}{2\mu_2}\,\frac{1}{p^2}\,\frac{\mu_1(\gamma^2 + a_1^2 p^2)^{\frac{1}{2}} - \mu_2(\gamma^2 + a_2^2 p^2)^{\frac{1}{2}}}{\mu_1(\gamma^2 + a_1^2 p^2)^{\frac{1}{2}} + \mu_2(\gamma^2 + a_2^2 p^2)^{\frac{1}{2}}}\,\frac{\exp[-(\gamma^2 + a_2^2 p^2)^{\frac{1}{2}} l_2]}{(\gamma^2 + a_2^2 p^2)^{\frac{1}{2}}}. \tag{17}$$

Substituting Eq. (16) into Eq. (11) and taking the inverse cosine transform, we obtain

$$\bar{w}_1 = \frac{2}{\pi}\int_0^\infty A_1(\gamma)\exp[(\gamma^2 + a_1^2 p^2)^{\frac{1}{2}} x]\cos\gamma y\, d\gamma. \tag{18}$$

An analogous equation can be written for $\bar{w}_2$.

The inversion of the one-sided Laplace transform can be carried out in a convenient manner by means of the Cagniard–de Hoop method. This method is discussed in Ref. [8]. In the next section we will need the displacement at the surface $y = 0$. Introducing a new variable ζ by the substitution

$$\gamma = p\zeta \tag{19}$$

we obtain for $y = 0$, $x > l_2$

$$\bar{w}_2 = (\bar{w}_2)_r + (\bar{w}_2)_d \tag{20}$$

where

$$(\bar{w}_2)_r = \frac{\tau_0}{\pi\mu_2}\,\frac{1}{p^2}\int_0^\infty \frac{\mu_1(\zeta^2 + a_1^2)^{\frac{1}{2}} - \mu_2(\zeta^2 + a_2^2)^{\frac{1}{2}}}{\mu_1(\zeta^2 + a_1^2)^{\frac{1}{2}} + \mu_2(\zeta^2 + a_2^2)^{\frac{1}{2}}}\,\frac{\exp[-p(\zeta^2 + a_2^2)^{\frac{1}{2}}(x + l_2)]}{(\zeta^2 + a_2^2)^{\frac{1}{2}}}\, d\zeta \tag{21}$$

and

$$(\bar{w}_2)_d = -\frac{\tau_0}{\pi\mu_2}\,\frac{1}{p^2}\int_0^\infty \frac{\exp[-p(\zeta^2 + a_2^2)^{\frac{1}{2}}(x - l_2)]}{(\zeta^2 + a_2^2)^{\frac{1}{2}}}\, d\zeta. \tag{22}$$

In Eq. (20), $(w_2)_d$ represents the displacement field due to the cylindrical wave directly emanating from the source point $x = l_2$, while $(w_2)_r$ represents the displacement field due to the reflected wave from the interface. Similarly, the Laplace transform of the displacement for $y = 0$, $x \leqq 0$ may be written as

$$\bar{w}_1 = (\bar{w}_1)_t = \frac{2\tau_0}{\pi}\,\frac{1}{p^2}\int_0^\infty \frac{\exp\{-p[(\zeta^2 + a_2^2)^{\frac{1}{2}} l_2 - (\zeta^2 + a_1^2)^{\frac{1}{2}} x]\}}{\mu_1(\zeta^2 + a_1^2)^{\frac{1}{2}} + \mu_2(\zeta^2 + a_2^2)^{\frac{1}{2}}}\, d\zeta \tag{23}$$

where $(w_1)_t$ is the displacement field due to the transmitted wave across the interface.

To obtain expressions for the displacement fields, it is found convenient to apply inverse Laplace transforms to the integrals in Eqs. (21), (22) and (23) and to the ratio $1/p^2$, separately, and then apply the convolution theorem. In using the Cagniard method to evaluate the inverse Laplace transforms to the integrals, a change of variables for ζ must be introduced in such a manner that the inverse Laplace transform of the integral over the new integration variable can be obtained by inspection, using the following property of the one-sided Laplace transform:

$$\mathscr{L}^{-1}\int_{t_1}^\infty e^{-pt} f(t)\, dt = f(t) H(t - t_1). \tag{24}$$

We will first consider the integral $(w_2)_d$, Eq. (22). The desired change of variable in the ζ-plane is defined by

$$(\zeta^2 + a_2^2)^{\frac{1}{2}}(x - l_2) = t. \tag{25}$$

By employing Eq. (25) and using Eq. (24), the inverse Laplace transform of the integral in Eq. (22) can easily be obtained. Subsequently using the convolution theorem we find for $t \geqq (x - l_2)a_2$

$$-\frac{\pi\mu_2}{\tau_0}(w_2)_d = \int_{(x-l_2)a_2}^{t} \frac{(t-u)du}{[u^2 - (x-l_2)^2 a_2^2]^{\frac{1}{2}}}$$

$$= t \ln \frac{t + [t^2 - (x-l_2)^2 a_2^2]^{\frac{1}{2}}}{(x-l_2)a_2} - [t^2 - (x-l_2)^2 a_2^2]^{\frac{1}{2}}. \tag{26}$$

As noted earlier $(w_2)_d(x, 0, t)$ for $x > l_2$ is the displacement field due to the cylindrical wave emanating directly from the applied load and it equals the solution in the case of a single material.

For the integral in $(\bar{w}_2)_r$, the appropriate change of variables is

$$(\zeta^2 + a_2^2)^{\frac{1}{2}}(x + l_2) = t. \tag{27}$$

Equation (21) then yields

$$\frac{\pi\mu_2}{\tau_0}(w_2)_r = \int_{(x+l_2)a_2}^{t} \frac{(t-u)F(u)du}{[u^2 - (x+l_2)^2 a_2^2]^{\frac{1}{2}}} \tag{28}$$

where

$$F(u) = \frac{\mu_1[u^2 + (x+l_2)^2(a_1^2 - a_2^2)]^{\frac{1}{2}} - \mu_2 u}{\mu_1[u^2 + (x+l_2)^2(a_1^2 - a_2^2)]^{\frac{1}{2}} + \mu_2 u}. \tag{29}$$

To evaluate the integral in Eq. (28) we consider an approximation for the function $F(u)$ which is valid when

$$c_2 t - (x + l_2)a_2 \ll 1. \tag{30}$$

This will be true for a point just behind the reflected wavefront. When this condition holds we may write

$$F \sim \frac{\mu_1 a_1 - \mu_2 a_2}{\mu_1 a_1 + \mu_2 a_2}, \tag{31}$$

and the integral in Eq. (28) may be evaluated to yield for $t > (x + l)a_2$

$$\frac{\pi\mu_2}{\tau_0}(w_2)_r = \frac{\mu_1 a_1 - \mu_2 a_2}{\mu_1 a_1 + \mu_2 a_2}\left\{t \ln \frac{t + [t^2 - (x+l_2)^2 a_2^2]^{\frac{1}{2}}}{(x+l_2)a_2} - [t^2 - (x+l_2)^2 a_2^2]^{\frac{1}{2}}\right\}. \tag{32}$$

This expression represents the wave motion reflected from the interface at $x = 0$. It can be noted that in the approximation implied by (30), the reflected wave has the form of a wave emanating from a concentrated line load at $x = -l_2$, with a magnitude modified by the constant given by Eq. (31).

It remains to investigate $\mathscr{L}^{-1}\{(\bar{w}_1)_t\}$ from Eq. (23). The appropirate change of variables is

$$(\zeta^2 + a_2^2)^{\frac{1}{2}} l_2 - (\zeta^2 + a_1^2)^{\frac{1}{2}} x = t. \tag{33}$$

Then Eq. (23) can be rewritten as

$$\frac{\pi}{2}\frac{(\bar{w}_1)_t}{\tau_0} = \frac{1}{p^2}\int_{a_2 l_2 - a_1 x}^{\infty} e^{-pt} G[\zeta(t)]\,dt \tag{34}$$

where

$$G(\zeta) = \frac{1}{\mu_1(\zeta^2 + a_1^2)^{\frac{1}{2}} + \mu_2(\zeta^2 + a_2^2)^{\frac{1}{2}}}\,\frac{(\zeta^2 + a_1^2)^{\frac{1}{2}}(\zeta^2 + a_2^2)^{\frac{1}{2}}}{(\zeta^2 + a_1^2)^{\frac{1}{2}} - x(\zeta^2 + a_2^2)^{\frac{1}{2}}}\,\frac{1}{\zeta}. \tag{35}$$

The inverse Laplace transform of Eq. (34) may be expressed in the form

$$\frac{\pi}{2}\frac{(w_1)_t}{\tau_0} = \int_{a_2 l_2 - a_1 x}^{t} (t-u) G(u)\,du. \tag{36}$$

To evaluate the integral in Eq. (36), we observe that the integrand $G(\zeta)$, given by Eq. (35), can be simplified for the case $\zeta \sim 0$ (this will be true for the point just behind the transmitted wave) to the following form:

$$G(\zeta) \sim \frac{a_1 a_2}{\mu_1 a_1 + \mu_2 a_2}\,\frac{\left[x^2 + l_2^2 - \left(\dfrac{a_2}{a_1} + \dfrac{a_1}{a_2}\right) x l_2\right]^{\frac{1}{2}}}{l_2 a_1 - x a_2}\,\frac{1}{[t^2 - (a_1 x - a_2 l_2)^2]^{\frac{1}{2}}}. \tag{37}$$

The equation (36) with Eq. (37) then yields

$$\begin{aligned}\frac{\pi}{2}\frac{(w_1)_t}{\tau_0} &\sim \frac{a_1 a_2}{\mu_1 a_1 + \mu_2 a_2}\,\frac{\left[x^2 + l^2 - \left(\dfrac{a_2}{a_1} + \dfrac{a_1}{a_2}\right) x l_2\right]^{\frac{1}{2}}}{l_2 a_1 - x a_2}\\ &\quad\times\left\{t \ln \frac{t + [t^2 - (a_1 x - a_2 l_2)^2]^{\frac{1}{2}}}{a_1 x - a_2 l_2} - [t^2 - (a_1 x - a_2 l_2)^2]^{\frac{1}{2}}\right\}.\end{aligned} \tag{38}$$

Equation (26), (32) and (38) thus give the displacement fields in the medium for the case when an anti-plane line load is applied at $x = l_2$. Similar expressions for the displacement fields may also be easily worked out when an anti-plane line load is applied at $x = -l_1$.

4. Dynamic stress field in the vicinity of the crack tip

Let us now return to the original problem namely the semi-infinite crack straddling the interface between the two elastic half-spaces, Fig. 1, and consider the displacement for $x \geqq a$. According to Eq. (1), the total displacement must vanish for $x \geqq a$. To obtain a vanishing displacement we will apply displacements equal and opposite to $(w_2)_d$, $(w_2)_r$ and $(w_2)_t$ which are defined in Eqs. (26), (32) and (38) (with the parameters suitably changed). These super-imposed displacements generate shear stresses at $y = 0$, $x \geqq a$, which will be computed in the sequel.

The wave propagation problems with $-(w_2)_d$, $-(w_2)_r$ and $-(w_2)_t$ applied in the region $x \geqq a$ are solved by a method which is based on the use of a Green's function. This method has been used by Kostrov [5] and Achenbach [6, 7] for problems of crack propagation. It is explained in considerable detail in Ref. [8]. If the surface $y = 0$ is subjected to a distribution of the form $(\tau_{yz})_2 = \tau(x, s)$, where $s = c_2 t$ the displacement field w_2 in the half plane $y \geqq 0$ is of the form (see Achenbach [8] p. 360):

$$w_2(x, y, s) = -\frac{1}{\pi\mu_2}\iint_S \frac{\tau(\bar{x}, \bar{s})}{R}\, d\bar{x}\, d\bar{s}. \tag{39}$$

In Eq. (39) S is that part of the $\bar{x}\bar{s}$-plane which falls inside the cone defined by

$$(s - \bar{s}) - [(x - \bar{x})^2 + y^2]^{\frac{1}{2}} \geqq 0, \quad s \geqq \bar{s} \geqq 0 \tag{40}$$

and

$$R^2 = (s - \bar{s})^2 - (x - \bar{x})^2 - y^2. \tag{41}$$

The region defined by Eq. (40) is a rather complicated area, being bounded in general by a hyperbola and a number of straight lines. For the case $y = 0$, the region of integration S reduces to a triangular region in the $\bar{x}\bar{s}$-plane and moreover the integrand in Eq. (39) simplifies significantly. The integral which is obtained from Eq. (39) by setting $y = 0$ can then be simplified considerably by introducing the following characteristic coordinates in the $\bar{x}\bar{s}$-plane:

$$\bar{\xi} = (\bar{s} - \bar{x})/\sqrt{2}, \quad \bar{\eta} = (\bar{s} + \bar{x})/\sqrt{2}. \tag{42}$$

By substituting Eq. (42) into Eq. (39) and setting $y = 0$, we find that the expression for the displacement field may be written as

$$w_2(\xi, \eta) = -\frac{1}{\pi\mu_2\sqrt{2}}\iint \frac{\tau(\bar{\xi}, \bar{\eta})}{(\xi - \bar{\xi})^{\frac{1}{2}}(\eta - \bar{\eta})^{\frac{1}{2}}}\, d\bar{\xi}\, d\bar{\eta}. \tag{43}$$

Equation (43) gives the desired relationship between the displacement and a distribution of surface tractions. If the stresses are known over a particular region, Eq. (43) will enable one to find the corresponding displacements and alternately, if the displacements are known over a given region, Eq. (43) will give an expression for the tractions on the same region. Thus for our case the displacement fields for $x \geqq a$ are given by Eqs. (26), (32) and (38) (with the parameters suitably changed) and the relevant stress fields can then be obtained by inverting Eq. (43).

Let us first consider the displacement field $(w_2)_d$ given by Eq. (26). As noted earlier, this displacement is due to the cylindrical wave emanating directly from the source point $x = l_2$. By setting up a new variable, $x' = x - l_2$ and employing the characteristic coordinates as defined by (42), the displacement field $(w_2)_d$ can be written as

$$(w_2)_d = -\frac{\tau_0 a_2}{(2)^{\frac{1}{2}}\pi\mu_2}\left\{(\xi + \eta)\ln\frac{(\xi)^{\frac{1}{2}} + (\eta)^{\frac{1}{2}}}{(\eta)^{\frac{1}{2}} - (\xi)^{\frac{1}{2}}} - 2(\xi)^{\frac{1}{2}}(\eta)^{\frac{1}{2}}\right\}. \tag{44}$$

Now consider a position defined by x_1, s_1 or ξ_1, η_1 in the region $x \geqq a$. At this point the displacement field will be equal and opposite to that given by Eq. (44). The relevant stress field can then be expressed as

$$\int_0^{\xi_1} \frac{d\bar{\xi}}{(\xi_1 - \bar{\xi})^{\frac{1}{2}}} \int_{\bar{\xi}+\sqrt{2}(a-l_2)}^{\eta_1} \frac{(\tau_{yz})_2^d(\bar{\xi}, \bar{\eta})}{(\eta_1 - \bar{\eta})^{\frac{1}{2}}} d\bar{\eta} = (2)^{\frac{1}{2}}\pi\mu_2(w_2)_d(\xi_1, \eta_1). \tag{45}$$

Equation (45) is an integral equation of the Abel type and can be inverted (see Appendix) to yield

$$(\tau_{yz})_2^d = -\frac{2\tau_0 a_2}{\pi[\eta - \xi - (2)^{\frac{1}{2}}(a - l_2)]^{\frac{1}{2}}} [(2)^{\frac{1}{2}}(a - l_2)]^{\frac{1}{2}} \frac{\xi}{\eta - \xi}. \tag{46}$$

In terms of the coordinates x and t, the stress is

$$(\tau_{yz})_2^d = -\frac{\tau_0}{\pi c_2(x - a)^{\frac{1}{2}}} (a - l_2)^{\frac{1}{2}} \frac{c_2 t - x + l_2}{x - l_2}. \tag{47}$$

In the vicinity of the crack tip, shear stress due to the cylindrical wave emanating directly from the source point $x = l_2$ thus follows from Eq. (47)

$$(\tau_{yz})_2^d = -K_d/(x - a)^{\frac{1}{2}} \tag{48}$$

where

$$K_d = \frac{\tau_0}{\pi c_2} \frac{c_2 t - a + l_2}{(a - l_2)^{\frac{1}{2}}}. \tag{49}$$

To compute the shear stress generated in the region due to the displacement field $-(w_2)_r$, we observe that the displacement field has the same analytical form due to a wave emanating from the point $x = -l_2$ but multiplied by a constant given by Eq. (31). This wave propagation problem is again similar to the one above and yields a shear stress field $(\tau_{yz})_2^r$ in the vicinity of the crack tip in the following form:

$$(\tau_{yz})_2^r = -K_r/(x - a)^{\frac{1}{2}} \tag{50}$$

where

$$K_r = \frac{\tau_0}{\pi c_2} \frac{\mu_2 c_1 - \mu_1 c_2}{\mu_2 c_1 + \mu_1 c_2} \frac{c_2 t - a - l_2}{(a + l_2)^{\frac{1}{2}}}. \tag{51}$$

It remains to investigate the shear stress $(\tau_{yz})_2^t$ in the vicinity of the crack tip due to the transmitted wave across the interface from material 1 to material 2. The displacement field in the medium 2 due to this transmitted wave follows from Eq. (38) and it is of the form

$$\frac{\pi}{2} \frac{(w_2)_t}{\tau_0} \sim -\frac{a_1 a_2}{\mu_1 a_1 + \mu_2 a_2} \frac{\left[x^2 + l_1^2 + \left(\frac{a_2}{a_1} + \frac{a_1}{a_2}\right)x l_1\right]^{\frac{1}{2}}}{l_1 a_2 + x a_1}$$
$$\times \left\{t \ln \frac{t + [t^2 - (a_2 x + a_1 l_1)^2]^{\frac{1}{2}}}{a_2 x + a_1 l_1} - [t^2 - (a_2 x + a_1 l_1)^2]^{\frac{1}{2}}\right\}. \tag{52}$$

By using a new variable $x' = x + l_1 a_1/a_2$ and employing the characteristic coordinates as defined by Eq. (42), the displacement field due to the transmitted wave can be expressed as

$$(w_2)_t \sim \frac{(2)^{\frac{1}{2}}\tau_0}{\pi} \frac{a_2^2}{\mu_1 a_1 + \mu_2 a_2} \frac{1}{\left[1 + \frac{(2)^{\frac{1}{2}} l_1}{\eta - \xi}\left(\frac{a_2}{a_1} - \frac{a_1}{a_2}\right)\right]^{\frac{1}{2}}}$$

$$\times \left\{(\xi + \eta) \ln \frac{(\xi)^{\frac{1}{2}} + (\eta)^{\frac{1}{2}}}{(\eta)^{\frac{1}{2}} - (\xi)^{\frac{1}{2}}} - 2(\xi)^{\frac{1}{2}}(\eta)^{\frac{1}{2}}\right\}. \tag{53}$$

To make the displacement field vanish ahead of the crack tip, we apply the equal and opposite displacement fields of Eq. (53). The relevant shear stress $(\tau_{yz})_2^t$ can then be obtained from Eq. (43). The inversion of the integral equation (43) for the transmitted wave part is very difficult because of the term

$$\left[1 + \frac{(2)^{\frac{1}{2}} l_1}{\eta - \xi}\left(\frac{a_2}{a_1} - \frac{a_1}{a_2}\right)\right]^{-\frac{1}{2}}$$

in Eq. (53). To get an insight of the effect of this transmitted wave on the stress field in the vicinity of the crack tip, we assume that $l_1 \ll a$. With this assumption, the terms inside the square bracket are expanded to yield a displacement field

$$(w_2)_t \sim -\frac{(2)^{\frac{1}{2}}\tau_0}{\pi} \frac{a_2^2}{\mu_1 a_1 + \mu_2 a_2}\left[1 - \frac{l_1}{(2)^{\frac{1}{2}}(\eta - \xi)}\left(\frac{a_2}{a_1} - \frac{a_1}{a_2}\right)\right]$$

$$\times \left\{(\xi + \eta) \ln \frac{(\xi)^{\frac{1}{2}} + (\eta)^{\frac{1}{2}}}{(\eta)^{\frac{1}{2}} - (\xi)^{\frac{1}{2}}} - 2(\xi)^{\frac{1}{2}}(\eta)^{\frac{1}{2}}\right\}. \tag{54}$$

The corresponding shear stress $(\tau_{yz})_2^t$ can then be obtained by following the analysis before and we find

$$(\tau_{yz})^t \sim \frac{\mu_2 \tau_0 a_2^2}{\pi[\mu_1 a_1 + \mu_2 a_2]} \frac{1}{\left[\eta - \xi - (2)^{\frac{1}{2}}\left(a + \frac{a_1}{a_2} l_1\right)\right]^{\frac{1}{2}}}$$

$$\times \left\{ 4\left[(2)^{\frac{1}{2}}\left(a + \frac{a_1}{a_2} l_1\right)\right]^{\frac{1}{2}} \frac{\xi}{\eta - \xi} \right.$$

$$\left. - (2)^{\frac{1}{2}} l_1 \left(\frac{a_2}{a_1} - \frac{a_1}{a_2}\right) \frac{\xi}{(\eta - \xi)^2} \frac{\left[\eta - \xi - 2(2)^{\frac{1}{2}}\left(a + \frac{a_1}{a_2} l_1\right)\right]}{\left[(2)^{\frac{1}{2}}\left(a + \frac{a_1}{a_2} l_1\right)\right]^{\frac{1}{2}}} \right\}. \tag{55}$$

In terms of the coordinates x and t, the expression (55) becomes

$$(\tau_{yz})_2^t \sim \frac{\mu_2\tau_0 c_1}{\pi c_2[\mu_1 c_2 + \mu_2 c_1]}\,\frac{1}{(x-a)^{\frac{1}{2}}}\left\{2\left(a+\frac{c_2}{c_1}l_1\right)^{\frac{1}{2}}\frac{c_2 t - x - \frac{c_2}{c_1}l_2}{x+\frac{c_2}{c_1}l_1}\right.$$

$$\left. -\frac{l_1}{2}\left(\frac{c_1}{c_2}-\frac{c_2}{c_1}\right)\frac{c_2 t - x - \frac{c_2}{c_1}l_1}{\left(x+\frac{c_2}{c_1}l_1\right)^2}\,\frac{x-2a-\frac{c_2}{c_1}l_1}{\left(a+\frac{c_2}{c_1}l_1\right)^{\frac{1}{2}}}\right\}. \tag{56}$$

In the vicinity of the crack tip, shear stress due to the transmitted wave from material 1 to material 2 thus follows from Eq. (56):

$$(\tau_{yz})_2^t = -\,K_t/(x-a)^{\frac{1}{2}} \tag{57}$$

where

$$K_t = -\frac{\mu_2\tau_0 c_1}{\pi c_2[\mu_1 c_2 + \mu_2 c_1]}\left\{2\frac{c_2 t - a - \frac{c_2}{c_1}l_1}{\left(a+\frac{c_2}{c_1}l_1\right)^{\frac{1}{2}}} + \frac{l_1}{2}\left(\frac{c_1}{c_2}-\frac{c_2}{c_1}\right)\frac{c_2 t - a - \frac{c_2}{c_1}l_1}{\left(a+\frac{c_2}{c_1}l_1\right)^{\frac{3}{2}}}\right\}. \tag{58}$$

It can be seen that Eq. (58) is similar to Eq. (48) for $\mu_1 = \mu_2$ as one would expect.

It is also to be noted that the expressions (50) and (57) are valid for points just behind the reflected and transmitted wave fronts respectively. To superimpose all the stress fields in the vicinity of the crack tip, they should arrive at the same time t to the crack tip. It is therefore assumed that l_1 and l_2 are so chosen that both the reflected and transmitted wavefronts reach the crack tip at the same time. The total shear field in the vicinity of the crack tip due to the anti-plane line loads at $x = -l_1$ and $x = l_2$ can then be written as

$$(\tau_{yz})_2 = -\,K/(x-a)^{\frac{1}{2}} \tag{59}$$

where

$$K = -\frac{\tau_0}{\pi c_2}\left\{\frac{c_2 t - a - l_2}{(a+l_2)^{\frac{1}{2}}} - \frac{c_2 t - a + l_2}{(a-l_2)^{\frac{1}{2}}} + \frac{\mu_2 c_1}{\mu_1 c_2 + \mu_2 c_1}\,\frac{l_2}{2}\,\frac{c_1}{c_2}\left(\frac{c_1}{c_2}-\frac{c_2}{c_1}\right)\right.$$

$$\left.\times\,\frac{c_2 t - a - l_2}{(a+l_2)^{\frac{3}{2}}}\right\}. \tag{60}$$

5. Conclusion

The intensity factor K is plotted versus $c_2 t/a$ for some values of l_2/a, μ_2/μ_1 and μ_1/μ_2 in Figs. 2 and 3 keeping $\rho_1 = \rho_2$. The figures indicate that the bimaterial interface has a sig-

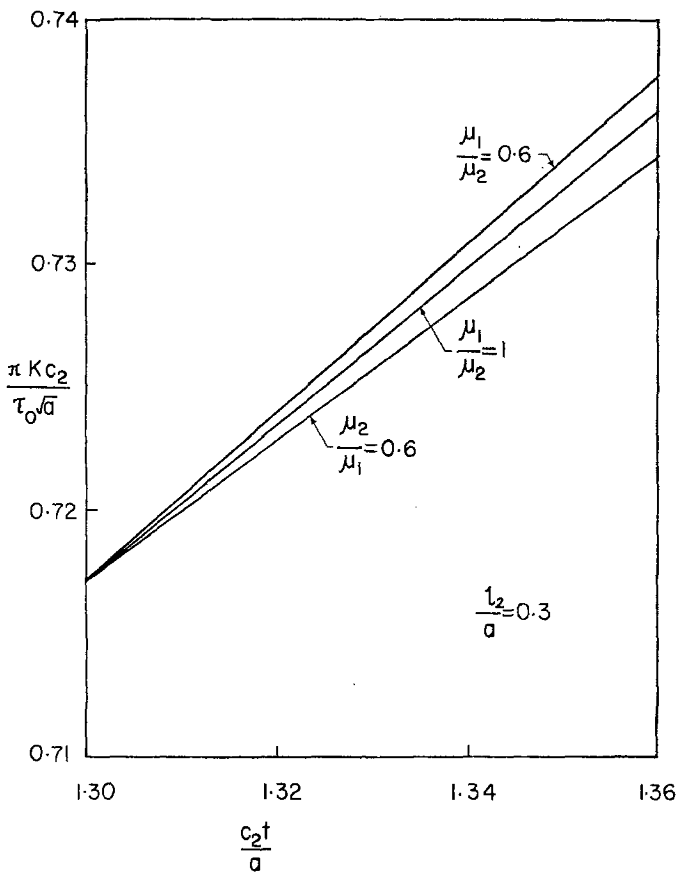

Figure 2. The stress intensity factor versus c_2t/a for $l_2/a = 0.3$.

nificant effect on the stress intensity factor at the crack tip. The contribution of the transmitted and reflected waves from the interface serves to decrease or increase the stress at the crack tip. The sign of the change depends critically on the material properties of the media. The stress intensity factor is decreased if the crack tip is in the softer material and vice versa. Thus by suitably choosing materials with proper ratios of material properties one can in principle control to a certain extent the stresses developed in the vicinity of the crack tip and thus reduce the chances of material failure.

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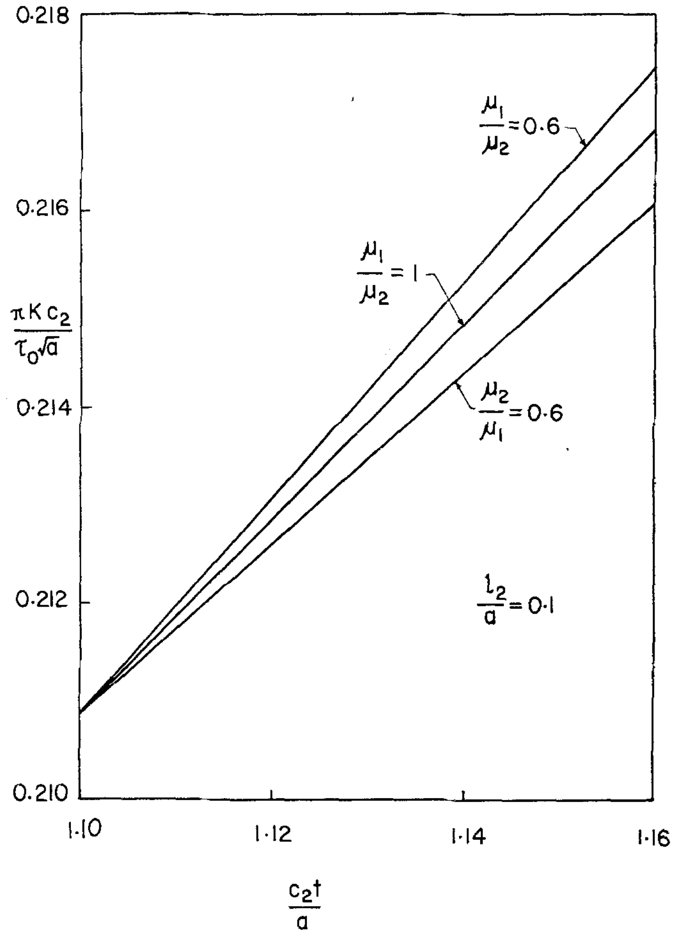

Figure 3. The stress intensity factor versus c_2t/a for $l_2/a = 0.1$.

Appendix

Consider an integral equation of the Abel type

$$\int_a^{u_1} \frac{du}{(u_1 - u)^{\frac{1}{2}}} \int_u^{v_1} \frac{\tau(u, v)}{(v_1 - v)^{\frac{1}{2}}} dv = f(u_1, v_1) \tag{A.1}$$

where $f(u_1, v_1)$ is a known function of u_1 and v_1 and $\tau(u, v)$ is the unknown function to be determined. This class of integral equations can be solved in closed form.

Multiplying both sides of (A.1) by $(u' - u_1)^{-\frac{1}{2}}$ and integrating over u_1, we have

$$\int_a^{u'} \frac{f(u_1, v_1)}{(u' - u_1)^{\frac{1}{2}}} du_1 = \int_a^{u'} \frac{du_1}{(u' - u_1)^{\frac{1}{2}}} \int_a^{u_1} \frac{du}{(u_1 - u)^{\frac{1}{2}}} \int_u^{v_1} \frac{\tau(u, v)}{(v_1 - v)^{\frac{1}{2}}} dv. \tag{A.2}$$

Let us define a function $h(u, v_1)$ as follows:

$$h(u, v_1) = \int_u^{v_1} \frac{\tau(u, v)}{(v_1 - v)^{\frac{1}{2}}} dv. \tag{A.3}$$

In view of (A.3), (A.2) can be written as

$$\int_a^{u'} \frac{f(u_1, v_1)}{(u' - u_1)^{\frac{1}{2}}} du_1 = \int_a^{u'} \frac{du_1}{(u' - u_1)^{\frac{1}{2}}} \int_a^{u_1} \frac{h(u, v_1)}{(u_1 - u)^{\frac{1}{2}}} du. \tag{A.4}$$

By interchanging orders of integration, we can write (A.4) in the form

$$\int_a^{u'} \frac{f(u_1, v_1)}{(u' - u_1)^{\frac{1}{2}}} du_1 = \int_a^{u'} h(u, v_1) du \int_u^{u'} \frac{du_1}{(u' - u_1)^{\frac{1}{2}}(u_1 - u)^{\frac{1}{2}}}. \tag{A.5}$$

The integral on u_1 at the right hand side can be evaluated (see Ref. [12]) and we find

$$\int_a^{u'} \frac{f(u_1, v_1)}{(u' - u_1)^{\frac{1}{2}}} du_1 = \pi \int_a^{u'} h(u, v_1) du. \tag{A.6}$$

We now differentiate (A.6) with respect to u' to get the relation between τ and f [see equation (A.3)]:

$$\frac{\partial}{\partial u'} \int_a^{u'} \frac{f(u_1, v_1)}{(u' - u_1)^{\frac{1}{2}}} du_1 = \pi \int_{u'}^{v_1} \frac{\tau(u', v)}{(v_1 - v)^{\frac{1}{2}}} dv. \tag{A.7}$$

Multiplying both sides of (A.7) by $(v' - v_1)^{-\frac{1}{2}}$ and integrating with respect to v_1, we have

$$\int_{u'}^{v'} \left[\frac{\partial}{\partial u'} \int_a^{u'} \frac{f(u_1, v_1)}{(u' - u_1)^{\frac{1}{2}}} du_1 \right] \frac{dv_1}{(v' - v_1)^{\frac{1}{2}}} = \pi \int_{u'}^{v'} \left[\int_{u'}^{v_1} \frac{\tau(u', v)}{(v_1 - v)^{\frac{1}{2}}} dv \right] \frac{dv_1}{(v' - v_1)^{\frac{1}{2}}}.$$

By interchanging orders of integration, we can write

$$\int_{u'}^{v'} \frac{dv_1}{(v' - v_1)^{\frac{1}{2}}} \left[\frac{\partial}{\partial u'} \int_a^{u'} \frac{f(u_1, v_1)}{(u' - u_1)^{\frac{1}{2}}} du_1 \right] = \pi \int_{u'}^{v'} \tau(u', v) dv \int_v^{v'} \frac{dv_1}{(v' - v_1)^{\frac{1}{2}}(v_1 - v)^{\frac{1}{2}}}$$

$$= \pi^2 \int_{u'}^{v'} \tau(u', v) dv. \tag{A.8}$$

Dropping the primes in (A.8) and differentiating both sides by v, we obtain the solution of the integral equation (A.1) in the following form

$$\tau(u, v) = \frac{1}{\pi^2} \frac{\partial}{\partial v} \int_u^v \left[\frac{\partial}{\partial u} \int_a^u \frac{f(u_1, v_1)}{(u - u_1)^{\frac{1}{2}}} du_1 \right] \frac{dv_1}{(v - v_1)^{\frac{1}{2}}}. \tag{A.9}$$

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